Universität Konstanz
Fachbereich Mathematik und Statistik
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Winter semester 2019/2020

Exercises for the course "Linear Algebra I"

Sheet 6

Hand in your solutions on Thursday, 5. Dezember 2019, 09:55, in the Postbox of your Tutor in F4. Please try to write your solutions clear and readable, write your name and the name of your tutor on each sheet, and staple the sheets together.

Exercise 6.1 *(5 points)*

Let K be a field and $n \in \mathbb{N}$.

(a) Show that there is a matrix $A \in M_{n\times n}(K)$ such that

$$\forall \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in M_{n\times 1}(K)\colon\ A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_n \\ \vdots \\ x_1 \end{pmatrix}.$$

(b) Let $A \in M_{n\times n}(K)$ be arbitrary and $i, j \in \{1, \dots, n\}$. Find matrices $B, C \in M_{n\times n}(K)$ such that $(BAC)_{ij} = A_{ij}$ and $(BAC)_{lm} = 0$ for all tuples $(l, m) \neq (i, j)$.

(c) Let $J \subseteq M_{n\times n}(K)$ be a set of $n \times n$-matrices over K with the following properties:
  (i) J contains a matrix $A \neq 0$;
  (ii) J is closed under addition, i.e., $A, B \in J \Rightarrow A + B \in J$;
  (iii) J is closed under left and right multiplication with elements from $M_{n\times n}(K)$, i.e., for all $A \in J$ and all $X \in M_{n\times n}(K)$ it applies $AX \in J$ and $XA \in J$.

  Show that $J = M_{n\times n}(K)$ holds.

Exercise 6.2 *(2 points)*

Let K be a field and V a K-vector space. Let U, W be subspaces of V such that
(i) $U + W = V$ and
(ii) $U \cap W = \{0\}$

Show that for each $v \in V$ there are uniquely determined elements $u \in U$ and $w \in W$ such that $v = u + w$.

Exercise 6.3 *(5 points)*

a) Let K be a field and let $(V, +_V)$, $(W, +_W)$ be two K-vector spaces with scalar multiplication $\cdot_V : K \times V \to V$ and $\cdot_W : K \times W \to W$ respectively. Let $U := V \times W$, let $+_U : U \times U \to U$ be defined by $(v_1, w_1) +_U (v_2, w_2) := (v_1 +_V v_2, w_1 +_W w_2)$ for $v_1, v_2 \in V$, $w_1, w_2 \in W$, and let $\cdot_U : K \times U \to U$ be defined by $\lambda \cdot_U (v, w) := (\lambda \cdot_V v, \lambda \cdot_W w)$ for $\lambda \in K$, $v \in V$, $w \in W$.

   Prove that $(U, +_U)$ with scalar multiplication $\cdot_U$ is a K-vector space.

b) In what follows F denotes the set of all mappings $f\colon \mathbb{R} \to \mathbb{R}$. We equip F with pointwise addition $(f + g)(x) := f(x) + g(x)$ and $\mathbb{R}$-scalar multiplication $\lambda \cdot f(x) := (\lambda f)(x)$ for $f, g \in F$ and $\lambda \in \mathbb{R}$. Let $F_1 := \{f \in F \mid \forall x \in \mathbb{R}\colon f(x) = f(-x)\}$ and $F_2 := \{f \in F \mid \forall x \in \mathbb{R}\colon f(x) = -f(-x)\}$. Show that F is an $\mathbb{R}$-vector space and that F_1 and F_2 are subspaces of F.

   Hint: Cf. Excercise 1.1(a)

c) Determine $F_1 \cap F_2$ and $F_1 + F_2$.

Exercise 6.4 *(4 points)*

Let X, Y be sets. We denote by $X \setminus Y$ the *difference set of X and Y*, i.e.,

$$X \setminus Y = \{x \in X : x \notin Y\}.$$

Now let $n \in \mathbb{N}$. Show that $\mathbb{Z}_n \setminus \mathbb{Z}_n^\times$ is a **commutative ring without** 1 if and only if $n = p^k$ for some prime number p and some $k \in \mathbb{N}_0$.